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Moment Spaces of Minimal Dimension*

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The objective of the present paper is to extend in several directions the result of [1]. For a given n, let F_n be the real linear space spanned by n linearly independent real valued bounded functions f_i, $i=1,..., n$, defined on some set X. Denote by $F_n \cdot F_n$ the real linear space spanned by the products $f_i f_j$, $i, j=1,..., n$. In the theory of design of experiments, the functions f_i are called regression functions. They induce a moment matrix (information matrix)

$$M(\zeta)=\{m_{ij}(\zeta)\}_{i,j=1}^{n}, \qquad m_{ij}(\zeta)=\int_X f_i(x) f_j(x)\, \zeta(dx)$$

where ζ is a given probability measure on X, called a design.

If for a given ζ, we treat $M(\zeta)$ as a point with coordinates $m_{ij}(\zeta)$, $i \leqslant j$, $i, j=1,..., n$ in $n(n+1)/2$, dimensional Eucledian space, then the set of all $M(\zeta)$, as ζ traverses the set Ξ of all probability measures, coincides with the moment space, say $\mathcal{M}_n$, [2], generated by the functions $f_i f_j$, $i, j=1,..., n$. The dimension of $\mathcal{M}_n$ is important for problems connected with the number of points of concentration of ζ. For example, it can be easily shown that if $\dim \mathcal{M}_n = s$, then for any given ζ there exists a design $\tilde{\zeta}$ such that $M(\tilde{\zeta})=M(\zeta)$ and $\tilde{\zeta}$ has no more than $s+1$ points of support. Since Ξ includes measures concentrated at one point of the set X, it is obvious that $\dim \mathcal{M}_n = \dim(F_n \cdot F_n)$. Thus we are naturally led to the problem of characterizing the linear spaces F_n which, for a given n, generate a space $F_n \cdot F_n$ of minimal dimension. This question is closely connected with the old Kiefer–Wolfowitz problem of describing those regression functions $f_1,..., f_n$ for which there exists an optimal design concentrated at n points [2].

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Our main result is contained in the following:

THEOREM. *In the above notation, let F_n be such that the linear space $F_n \cdot F_n$ has no zero divisor. Then* $\dim(F_n \cdot F_n) \geqslant 2n-1$, *with equality only if there exists a basis of F_n formed by n functions of the form ωq^{i-1}, $i=1,..., n$, where the functions ω and q satisfy the following conditions*:

(i) *$\omega(x)$ is a bounded function, $|q(x)|<\infty$ for all $x \in X$, such that $\omega(x) \neq 0$, and it is possible that there exist points $x \in X$, at which $\omega(x)=0$ and $q(x)=\infty$. In the latter case the functions ω and q are defined in such a way that $(\omega q^{i-1})(x)=0$, $i=1,..., n-1$, and $(\omega q^{n-1})(x) \neq 0$;*

(ii) *There exist $4n-3$ distinct points $x_j \in X$, $j=1,..., 4n-3$ such that $q(x_j) \neq q(x_i)$, $i \neq j$, $i, j=1,..., 4n-3$, and for at least $4n-4$ of these points $\omega(x_j) \neq 0$. And conversely, if the space F_n is spanned by the functions of the above form then the linear space $F_n \cdot F_n$ has no zero divisor and* $\dim(F_n \cdot F_n)=2n-1$.

Proof. In what follows, by a polynomial in F_n (resp. in $F_n \cdot F_n$) we mean a linear combination of basis functions of F_n (resp. $F_n \cdot F_n$) and use the letters α, β to denote the coefficients of these polynomials. We agree also that the equality $f=g$ means, unless stated explicitly otherwise, that $f(x)=g(x)$, for all $x \in X$. We shall repeatedly exploit the assumption of the theorem that the space $F_n \cdot F_n$ has no zero divisor. This means that if $PQ=0$, for some functions $P \in F_n \cdot F_n$, $Q \in F_n \cdot F_n$, then at least one of the above two functions is equal to zero. Observe first that from this condition, it follows that the space F_n also has no zero divisor. In fact, let f, g be any two elements of F_n, such that $fg=0$. Then $f^2 g^2=0$. But f^2, $g^2 \in F_n \cdot F_n$, from which it follows that at least one of the functions f^2, g^2 is a zero function. Consequently, the same is true for the functions f, g. For the sake of brevity we shall refer to this property of the spaces F_n and $F_n \cdot F_n$ as (*).

Sufficiency. Let the functions ωq^{i-1}, $i=1,..., n$ constitute a basis of the space F_n and let $x_j \in X$, $j=1,..., 4n-3$ be the points at which the functions ω and q satisfy condition (ii) of the theorem. Consider first any $2n-1$ of the above points, say $x_1,..., x_{2n-1}$, such that $\omega(x_j) \neq 0$, $j=1,..., 2n-1$. Observe that, according to condition (i) of the theorem, $|q(x_j)|<\infty$, $j=1,..., 2n-1$. Then $\det\{\omega^2(x_j)\, q^{i-1}(x_j)\}_{i,j=1}^{2n-1} \neq 0$, which implies that the functions $\omega^2 q^{i-1}$, $i=1,..., 2n-1$, spanning the space $F_n \cdot F_n$ are linearly independent on X. Hence $\dim(F_n \cdot F_n)=2n-1$ and it is easy to see that condition (i) of the theorem guarantees the boundness of all functions belonging to F_n. Now it remains to show that conditions (i) and (ii) imply condition (*) for the space $F_n \cdot F_n$. Consider two polynomials:

$$P = \omega^2 \sum_{i=1}^{2n-1} \alpha_i q^{i-1} \in F_n \cdot F_n$$

and

$$Q = \omega^2 \sum_{i=1}^{2n-1} \beta_i q^{i-1} \in F_n \cdot F_n$$

such that $PQ = 0$. Then $(PQ)(x_j) = 0$, $j = 1,..., 4n-3$, from which it follows that one of the polynomials, say P, has more than $2n-2$ distinct zeros among the above points x_j, $j = 1,..., 4n-3$. If all these zeros x_j of P are such that $\omega(x_j) \neq 0$, and, consequently, in view of condition (i), $|q(x_j)| < \infty$, then the condition $q(x_i) \neq q(x_j)$, $i \neq j$ immediately implies that $P = 0$. According to conditions (i) and (ii) of the theorem, it is possible that among the points x_j, $j = 1,..., 4n-3$ there is only one point at which $|q|$ is infinite. If such a point is a zero of P, then condition (i) of the theorem implies that $\alpha_{2n-1} = 0$. Hence, in this case P is a polynomial in q of degree $2n-3$ having more than $2n-3$ distinct zeros $q_j = q(x_j)$, $|q_j| < \infty$. Consequently, again $P = 0$. This establishes (*) and completes the proof of sufficiency.

Necessity. The idea of the proof is based on constructing a basis in F_n formed by fundamental Lagrange polynomials. Let x_j, $j = 1,..., n$ be any n distinct points in X chosen so that $\det\{f_i(x_j)\}_{i,j=1}^{n} \neq 0$. Such a choice is possible since the functions f_i, $i = 1,..., n$ are assumed to be linearly independent on X. This guarantees the existence of n fundamental Lagrange polynomials $p_i \in F_n$, $i = 1,..., n$ induced by the points $x_1,..., x_n$, i.e., polynomials determined by the requirements

$$p_i(x_j) = \delta_{ij}, \qquad i, j = 1,..., n. \tag{1}$$

Obviously the functions p_i, $i = 1,..., n$ are linearly independent on X, so they form a basis of F_n. Thus the space $F_n \cdot F_n$ is spanned by the functions $p_i p_j$, $i, j = 1,..., n$. Consider now the following system of $2n-1$ functions in $F_n \cdot F_n$:

$$p_1 p_j, \quad j = 2,..., n; \qquad p_i^2, \quad i = 1,..., n. \tag{2}$$

From (1) we have

$$\left(p_1 \sum_{j=2}^{n} \alpha_j p_j + \sum_{i=1}^{n} \beta_i p_i^2 \right)(x_l) = \beta_l, \qquad l = 1,..., n. \tag{3}$$

(3) and condition (*) for the space F_n imply that the equation

$$p_1 \sum_{j=2}^{n} \alpha_j p_j + \sum_{i=1}^{n} \beta_i p_i^2 = 0$$

holds only if $\alpha_j = 0$, $j = 2, ..., n$; $\beta_i = 0$, $i = 1, ..., n$. This means that the functions (2) are linearly independent on X and shows that $\dim(F_n \cdot F_n) \geqslant 2n - 1$.

According to the necessary condition of the theorem, $\dim(F_n \cdot F_n) = 2n - 1$. Therefore the system of functions (2), forms a basis of $F_n \cdot F_n$. Consequently,

$$p_k p_l = p_1 \sum_{i=2}^{n} \alpha_{i1}^{(k,l)} p_i + \sum_{i=1}^{n} \beta_{i1}^{(k,l)} p_i^2, \qquad k, l = 1, ..., n. \tag{4}$$

But according to (1), for any $k \neq l$, $(p_k p_l)(x_j) = 0$, $j = 1, ..., n$, which implies that in (4), $\beta_{i1}^{(k,l)} = 0$, $i = 1, ..., n$, $k \neq l$. Thus we have

$$p_k p_l = p_1 \sum_{i=2}^{n} \alpha_{i1}^{(k,l)} p_i, \qquad k \neq l,\ k, l = 1, ..., n \tag{5}$$

and similarly

$$p_k p_s = p_1 \sum_{i=2}^{n} \alpha_{i1}^{(k,s)} p_i, \qquad k \neq s,\ k, s = 1, ..., n.$$

By virtue of (*) for the spaces $F_n \cdot F_n$ and F_n we obtain from the above two equalities that

$$p_l \sum_{i=2}^{n} \alpha_{i1}^{(k,s)} p_i - p_s \sum_{i=2}^{n} \alpha_{i1}^{(k,l)} p_i = 0,$$

$$s \neq k, \qquad s \neq l, \qquad k \neq l, \qquad l, s, k = 1, ..., n.$$

Applying the last equality for $x = x_s$, gives

$$\alpha_{s1}^{(k,l)} = 0, \quad s = 2, ..., n, \quad s \neq k; \qquad s \neq l; \qquad k \neq l.$$

Consequently, (5) takes the form

$$p_k p_l = p_1 (\alpha_{k1}^{(k,l)} p_k + \alpha_{l1}^{(k,l)} p_l), \qquad k \neq l,\ k, l = 1, ..., n.$$

Changing p_1 on p_s we finally obtain that the functions p_i must satisfy the following relationships:

$$p_k p_l = p_s (\alpha_{ks}^{(k,l)} p_k + \alpha_{ls}^{(k,l)} p_l), \qquad k \neq l,\ k, l, s = 1, ..., n. \tag{6}$$

It is important to notice that in (6) $\alpha_{ks}^{(k,l)} \neq 0$, $\alpha_{ls}^{(k,l)} \neq 0$, $s \neq k, l$, due to (*) for the space F_n. The relationship (6) will serve as the main tool for proving the necessity of the conditions of the theorem. Let one of the functions p_i, $i = 1,..., n$, say p_s, vanish at some point $x \in X$. Then we find from (6) that $(p_k p_l)(x) = 0$, $k \neq l$, $k, l = 1,..., n$. This means that a zero of one of the above n functions p_i is necessarily a zero of all but at most one of the remaining functions p_i. Due to this fact, all the functions $p_i p_j$, $i \neq j$, $i, j = 1,..., n$, vanish at the same points $x \in X$. Thus the set $X_1 = \{x \in X: (p_i p_j)(x) = 0\}$ is the same for all $i \neq j$, $i, j = 1,..., n$, and, consequently, $X \setminus X_1 = \{x \in X: p_i(x) \neq 0, i = 1,..., n\}$. Now we find from (6) that

$$(\alpha_{ks}^{(k,l)} p_k + \alpha_{ls}^{(k,l)} p_l)(x) \neq 0, \qquad x \in X \setminus X_1,\ k \neq l,\ k, l, s = 1,..., n.$$

(Observe that, by virtue of (*) for the space F_n, $X_1 \neq X$.)
Accordingly, (6) implies, for fixed k, l, $k \neq l$, that

$$p_s(x) = \frac{p_k p_l}{\alpha_{ks}^{(k,l)} p_k + \alpha_{ls}^{(k,l)} p_l}(x), \qquad s = 1,..., n,\ x \in X \setminus X_1.$$

We have thus established that on the subset $X \setminus X_1$ the space F_n is spanned by the following n functions:

$$\frac{p_l}{\Delta_s}, \qquad s = 1,..., n \tag{7}$$

Here $\Delta_s = \alpha_{ks}^{(k,l)} + \alpha_{ls}^{(k,l)} q$, $s = 1,..., n$, $q = p_l/p_k$, and for $x \in X \setminus X_1$, $\infty \neq |q(x)| \neq 0$, $\Delta_l(x) = 1$, $\Delta_k(x) = q(x)$, $\Delta_s(x) \neq 0$, $s = 1,..., n$. Again applying (*) for the space F_n, we obtain from (6) that

$$\frac{\alpha_{ki}^{(k,l)}}{\alpha_{kj}^{(k,l)}} \neq \frac{\alpha_{li}^{(k,l)}}{\alpha_{lj}^{(k,l)}}, \qquad i \neq j,\ i, j = 1,..., n,\ k \neq l.$$

In view of this, together with the fact that $\alpha_{ls}^{(k,l)} \neq 0$, $s = 1,..., n$, $s \neq k$, we find that $\prod_{s=1}^n \Delta_s$ is a polynomial in q of degree $n-1$, having $n-1$ distinct zeros. This allows us to conclude that any fraction of the form $(\prod_{s=1}^n \Delta_s)^{-1} q^{i-1}$, $i = 1,..., n$ is a linear combination of the n fractions Δ_s^{-1}, $s = 1,..., n$, and it is obvious that the converse assertion is also true. So we obtain from (7) that the functions ωq^{i-1}, $i = i,..., n$, where $\omega = (\prod_{s=1}^n \Delta_s)^{-1} p_l$, span the space F_n on the subset $X \setminus X_1$. Summarizing the previous discussion, we find that the functions

$$g_i(x) = \begin{cases} (\omega q^{i-1})(x), & x \in X \setminus X_1 \\ h_i(x), & x \in X_1 \end{cases} \qquad i = 1,..., n \tag{8}$$

where each function h_i is a linear combination of the functions $p_1,..., p_n$, span the space F_n. Now consider $0 \neq P = p_1 p_2 \in F_n \cdot F_n$ and any $Q \in F_n \cdot F_n$, such that $Q(x) = 0$, $x \in X \backslash X_1$. Then $PQ = 0$, since $(p_1 p_2)(x) = 0$, $x \in X_1$. Thus the condition (*) for the space $F_n \cdot F_n$ implies that $Q = 0$. This means that $2n-1$ basis functions (2) of the space $F_n \cdot F_n$ also constitute a basis of this space on the subset $X \backslash X_1$. But, according to (8), the space $F_n \cdot F_n$ on $X \backslash X_1$ is spanned by $2n-1$ functions ωq^{i-1}, $i = 1,..., 2n-1$. Thus we deduce that the functions ωq^{i-1}, $i = 1,..., 2n-1$ are linearly independent on $X \backslash X_1$. Now we determine the form of the functions h_i, $i = 1,..., n$ in (8). From (8) we see that, for any fixed s, $2 \leqslant s \leqslant 2n$, the functions $g_i g_j$, $i, j = 1, 2,..., n$, $i + j = s$, coincide on the subset $X \backslash X_1$: $(g_i g_j)(x) = (\omega^2 q^{s-2})(x)$, $x \in X \backslash X_1$. Furthermore, the function $\omega^2 q^{s-2}$ is not identically zero on $X \backslash X_1$, since it is one of the basis functions of the space $F_n \cdot F_n$ on this set. Hence, for any given s, $2 \leqslant s \leqslant 2n$, the functions $h_i h_j$, $i, j = i,..., n$, $i + j = s$, must coincide on X_1, since otherwise at least two of the functions $g_i g_j$, $i, j = 1,..., n$, $i + j = s$, will be linearly independent on X, and, consequently, $\dim(F_n \cdot F_n) > 2n-1$. From the previously established relationships between the functions $h_i h_j$ it is easy to see that a zero of one of the n functions h_i, $i = 1,..., n$ is necessarily a zero of all other functions h_i with the possible exception of one of the two functions h_1 and h_n. Thus, at the points $x \in X_1$ where $h_1(x) \neq 0$, the functions h_i, $i = 1,..., n$ are of the form stated in the theorem, with $\omega = h_1$ and $q = h_2/h_1$, and at all other points $x \in X_1$ (if such points exist) $h_i(x) = 0$, $i = 1,..., n-1$. Now consider these latter points $x \in X_1$. We can define $\omega(x) = 0$, $q(x) = 0$, if $h_n(x) = 0$. Otherwise the functions q and ω can be defined at the above points according to condition (i) of the theorem.

Thus we have proved the existence of a basis of the space F_n formed by functions of the form ωq^{i-1}, $i = i,..., n$. The necessity of the other part of condition (i) concerning the functions q and ω follows immediately from the assumed boundness of the functions f_i, $i = 1,..., n$, spanning F_n. Next we will show that condition (ii) of the theorem is a consequence of condition (*) for the space $F_n \cdot F_n$. We divide the set X into three disjoint subsets: $X_{01} = \{x \in X\colon \omega(x) = 0,\ |q(x)| \neq \infty\}$, $X_{02} = \{x \in X\colon \omega(x) = 0,\ |q(x)| = \infty\}$ and $X \backslash X_0 = \{x \in X\colon \omega(x) \neq 0\}$, where $X_0 = X_{01} \cup X_{02}$. Let the function q take only s distinct values, $q_1,..., q_s$, on the set $X \backslash X_0$, where, according to condition (i), $|q_i| < \infty$, $i = 1,..., s$. Consider $P = \omega^2 \prod_{i=1}^{[s/2]} (q - q_i)$ and $Q = \omega^2 \prod_{i=[s/2]+1}^{s} (q - q_i)$, where ω and q satisfy condition (i) of the theorem and $[\cdot]$ denotes the integer part of a number. If the set X_{02} is empty then, for $s < 4n-3$, $P, Q \in F_n \cdot F_n$, and $PQ = 0$. However, $P \neq 0$ and $Q \neq 0$ since $\omega(x) \neq 0$, $x \in X \backslash X_0$. In the case where the set X_{02} is not empty the same is true for $s < 4n-4$, since in this case the polynomial P is of degree less than $2n-2$, so that due to (i)(*), $P(x) = 0$, $x \in X_{02}$. Hence, in both cases condition (*) fails for the space $F_n \cdot F_n$. This completes the proof of the theorem.

Remarks. (i) Condition (*) cannot be eliminated. Let $X = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = \varnothing$ and let f_i be a characteristic function of the set Y_i, $i = 1, 2$. Then $\dim(F_2 \cdot F_2) = 2 < 3$, since $f_1 f_2 = 0$.

(ii) If condition (*) holds for F_n it does not necessarily hold for $F_n \cdot F_n$. For example, let X consist of $2n-1$ distinct points x_j, $j = 1, ..., 2n-1$ and $f_i(x) = x^{i-1}$, $i = 1, ..., n$. Then $\sum_{i=1}^{n} \alpha_i f_i \cdot \sum_{i=1}^{n} \beta_i f_i = 0$ implies that one of the two polynomials on the left-hand side has more than $n-1$ distinct zeros, and consequently, it is zero. This means that in the considered case F_n has no zero divisor. On the other hand, $F_n \cdot F_n$ is spanned by the functions x^{i-1}, $i = 1, ..., 2n-1$. Thus, taking, e.g., $P = (x - x_1) \cdots (x - x_{2n-2}) \in F_n \cdot F_n$ and $Q = x - x_{2n-1} \in F_n \cdot F_n$ we see that $PQ = 0$, while $P \neq 0$, $Q \neq 0$.

(iii) The theorem is valid also for the case when F_n is a linear space spanned by complex valued functions over a field of complex numbers.

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The following corollary generalizing the result of [1] can be easily obtained from the theorem:

COROLLARY. *In the above notation, let X contain at least* $4n-3$ *distinct points. Then the space* $F_n \cdot F_n$ *is spanned by* $2n-1$ *functions, forming a Chebyshev system of minimal degree* $2n-1$ *on X if and only if there exists in* F_n *a basis formed by the functions of the form* ωq^{i-1}, $i = 1, ..., n$, *where q is such that* $q(x) \neq q(y)$, $x \neq y$, $x, y \in X$ *and* $\omega(x) \neq 0$, $x \in X$.

Proof. The sufficiency is obvious. For proving the necessity show first that under the assumptions of the corollary the space $F_n \cdot F_n$ satisfies condition (*). Fix in the space $F_n \cdot F_n$ a basis of $2n-1$ functions forming a Chebyshev system of degree $2n-1$, and let P, Q be any two polynomials in $F_n \cdot F_n$ (with respect to the above basis), such that $PQ = 0$. Since the set X contains more than $4n-4$ distinct points, one of the above two polynomials has more than $2n-2$ distinct zeros, and, consequently, it equals zero, due to the assumed Chebyshev property of the basis functions of $F_n \cdot F_n$. Hence, all conditions of the theorem are fulfilled, which implies the existence in F_n a basis formed by the functions of the form ωq^{i-1}, $i = 1, ..., n$. Due to the Chebyshev property of these functions, the functions ω and q must be as in the corollary.

In conclusion, we present two examples of linear spaces of functions F_n, generating a linear space $F_n \cdot F_n$ of minimal dimension.

EXAMPLE 1. The system of rational functions. Let $f_i = (x - \alpha_i)^{-1}$, $x \in$

$[a, b]$, $\alpha_i \notin [a, b]$, $i = 1,..., n$. Denote $\omega(x) = \prod_{i=1}^{n} (x - \alpha_i)^{-1}$. Then we find that F_n is spanned by the functions ωx^{i-1}, $i = 1,..., n$.

EXAMPLE 2. The system of trigonometric functions: 1, $\sin x$, $\cos x,..., \sin mx$, $\cos mx$, $x \in R$, m is a fixed integer. Since $\sin kx = (e^{ikx} + e^{-ikx})/2i$, $\cos kx = (e^{ikx} + e^{-ikx})/2$, $k = 1, 2,..., m$, we immediately obtain that in the case considered, F_n, $n = 2m + 1$, is spanned by the functions e^{-ikx}, $k = -m, -m + 1,..., m$. See Remark (iii).

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